Europ. J. Combinatorics (1991) **12**, 23–31

Polarized Spaces of Polar Rank Three Which are Locally a Direct Product

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We present characterizations of geometries associated with buildings of type $C_{n,n-2}$, $D_{n,n-2}$ $(n \geq 4)$, $E_{7,4}$, $E_{8,5}$.

1. Definitions and Notations

An incidence system $\Gamma = (\mathscr{P}, \mathscr{L})$ is a set $\mathscr{P}$ of points together with a family $\mathscr{L}$ of distinguished subsets of $\mathscr{P}$ of cardinality at least 2, called lines. Two points p and q are said to be collinear if they are on a common line (denoted by $p \perp q$).

The incidence system Γ is said to be (partial) linear if any two distinct points are on at most one line. If p and q are two collinear, distinct points of a linear incidence system, then pq stands for the unique line through p and q.

The collinearity graph of Γ is the graph the vertices of which are the points of Γ and the edges of which are the pairs of distinct collinear points. Terms such as connectedness, path and distance will be used in Γ when they are meant for the collinearity graph. For points p and q, $d(p, q)$ denotes the distance between p and q. Further, $p^\perp$ stands for the set of all points collinear with p. If X is a set of points, then $X^\perp = \bigcap_{p \in X} p^\perp$.

A subspace X of Γ is a subset of $\mathscr{P}$ such that every line intersecting X in two distinct points is completely contained in X. A subspace is singular if all its points are pairwise collinear. A singular subspace is called maximal if it is not properly contained in another singular subspace. The length i of a longest chain $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_1 = X$ of singular subspaces X_j of a singular subspace X is called the rank of X. A maximal singular subspace of rank at least 2 will be called a max space for short.

A geodesic is a path joining p to q, the length of which is equal to the distance $d(p, q)$. A set of points X is called geodesically closed or convex if for every pair of points p and q of X, every geodesic joining p to q is contained in X. Now it is clear that the intersection of geodesically closed sets is again geodesically closed. Moreover, any intersection of geodesically closed subspaces is also a geodesically closed subspace. This allows us to talk about the geodesical closure of a set of points X, denoted by $\langle X \rangle$.

We recall from [5] that Γ is a polar space of finite rank r if the following conditions are satisfied: (i) for every point p and every line L, p is collinear with either one or all points of L; (ii) no point of Γ is collinear with all other points; (iii) every singular subspace has rank at most r, and there is such a singular subspace of rank r.

A polar space of rank 2 is also called a generalized quadrangle.

The results of Buekenhout and Shult [5], Tits [12] and Veldkamp [14] classify all polar spaces of rank r at least 3, and having lines containing at least three points: these are exactly the point–line geometries $C_{r,1}$ and $D_{r,1}$ associated to the buildings of type C_r and D_r. Polar spaces of type $D_{r,1}$ will be called lean polar spaces, and polar spaces not of this type will be called fat polar spaces.

We will say that Γ is a polarized space of polar rank $r + 1 \geq 3$ if the following conditions hold:

(1) If L is a line and p is a point collinear with at least two distinct points of L, then p is collinear with all points of L.

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(2) If p and q are non-collinear points with $|p^{\perp} \cap q^{\perp}| \geqslant 2$, then $p^{\perp} \cap q^{\perp}$ is a polar space of rank r.
(3) (i) The structure is connected but is not complete; (ii) every line contains at least three points; (iii) the set $p^{\perp} - \{p\}$ is connected for every point p; (iv) the maximal singular subspaces have finite rank.

We will mainly consider polarized spaces of polar rank 3.

Two points p and q at distance 2 form a polar pair or a special pair according to whether or not the condition of axiom 2 holds.

The geodesic closure of a polar pair is isomorphic to a polar space [4] and is called a hyperline or a symplectum.

Let p be a point of Γ. The residue of Γ at p, denoted by Γ_p, is the incidence system $(\mathscr{P}_p, \mathscr{L}_p)$ defined as follows: $\mathscr{P}_p$ is the set of all lines of Γ on p; the set of all lines on p contained in a plane (singular subspace of rank 3) of Γ on p is an element of $\mathscr{L}_p$. If A is an incidence system isomorphic to Γ_p for all points p, then Γ is said to be locally A. Remark that axiom 3(iii) states that Γ_p is connected for every point p. We will say that Γ is locally connected for every point p.

Two lines L and M are called coplanar if they are contained in some plane. Then every point of L is collinear with every point of M, and so we write $L \perp M$. We remark that this notation keeps its right meaning in Γ_p, if we use $\perp$ as collinearity in Γ_p as well.

If X_p is a subset of Γ_p, then the union of all lines in X_p is denoted by $\cup X_p$. Hence $\cup \mathrm{X_p}$ is a subset of the point set of Γ.

We recall a result from [4].

Proposition *If Γ is a polarized space of polar rank* 3, *then for every point p the following properties hold in Γ_p:*
(i) two distinct collinear points are contained in exactly one line;
(ii) two points at distance 2 *are contained in a geodesically closed subspace which, together with its lines, is a generalized quadrangle (these generalized quadrangles will be called quads—a quad which is not a grid is said to be fat);*
(iii) the structure is not complete and is connected.

For definitions such as diagram, geometry, truncation, etc. we refer to [2].

2. Characterization of $C_{n,n-2}$

Suppose that each line X of the polarized space Γ is contained in a hyperline of type C_3 and a max space, partioning the set of planes on X. Then Γ is the geometry of points and lines corresponding to a thick building of type $C_{n,n-2}$ for $n \geqslant 4$ or a quotient of it.

Proof. First consider the residue Γ_p at a point p: it will be connected in view of axiom 3(i). By supposition there is on each point x of Γ_p exactly one fat quad and one maximal singular space of rank at least 2, such that they form a partition of all lines through x.

In Γ_p we can distinguish between two kinds of lines; (i) lines contained in a fat quad, and (ii) lines not contained in any fat quad.

We will call the latter special lines, and the maximal singular spaces having no line in common with a fat quad are special spaces. It is clear that there is exactly one special space on a given special line.

We will prove that Γ_p can be identified with the direct product of a projective space and a fat generalized quadrangle (at least three lines on a point), and hence is of type $A_i \times C_2$.

It should be clear that two fat quads are necessarily disjoint, and the same can be said about the special spaces. From the proof of Lemma 3.5 in [10] we take over the following property: every point not contained in a given fat quad, H, is collinear with exactly one point of H. In particular, it follows that each special space R and each fat quad H meet in one point: for suppose that $R \cap H = \emptyset$, then we can choose points r in R and x in H such that $r \perp x$. The line rx is not contained in the fat quad H on x, so must be special. But then $rx \subset R$ because there is only one special space on r. This gives us $x \in R \cap H$, contradicting our assumption. On the other hand, suppose x and y would be two distinct points of R and H. Then R and H would have the line xy in common, again a contradiction.

Consider now two fat quads, H and H'. For a point x of H, we define $\sigma(x) = x'$ as the unique meeting point of H' and the special space on x. Clearly σ is a map, and interchanging the roles of H and H' we see that σ is a bijection from H to H'. We show that σ is an isomorphism. consider therefore two collinear points x and y of H; put $x' = \sigma(x)$ and $y' = \sigma(y)$. Let Q be the grid on x, y and x'; then Q has x' in common with H. This means that $Q \cap H'$ is a line L of H', so there exists a point y'' (on L) collinear with y. But y is collinear with only one point of H', so $y'' = y'$. In particular, $y' \perp x'$, so σ is an isomorphism.

Next consider two special spaces, R and R'. For a point x of R, we define $\tau(x) = x'$ as the unique meeting point of R' and the fat quad on x. As before, τ is a bijection. We prove that τ maps lines of R onto lines of R'. Let L be any line of R, and choose a point x on L. Suppose first that $\tau(x)$ is collinear with x. Call Q the grid containing L and x'; then Q has a line L' in common with R. If y is a point of L, then $\tau(y)$ will be a point of L'. So far τ maps lines onto lines. Suppose now that $\tau(x)$ is at distance 2 of x. Choose $x'' \in \tau(x)^\perp \cap x^\perp$, and let R'' be the special space on x''. By the first part we can define an isomorphism τ' from R to R'', and an isomorphism τ'' from R'' to R'. So $\tau = \tau''\ \tau'$ is also an isomorphism.

We conclude that Γ_p can be identified with the direct product of a projective space and a (fat) generalized quadrangle as follows. For any special space R and fat quad H define the canonical map:

$$\rho\colon\quad \Gamma_p \to R \times H, \qquad x \to (\tau(x), \sigma(x)).$$

Going back to Γ, we make the following observations: every line of Γ is contained in exactly one special space. If two lines meet in a point p, then our knowledge about Γ_p tells us that the corresponding special spaces are isomorphic; in particular, they have the same dimension. In view of the connectedness of Γ, this is so for all special spaces of Γ. An analogous reasoning leads us to say that all fat hyperlines of Γ are isomorphic. We may take $\Gamma_p \simeq A_i(K) \times C_2(K)$ independent of the point p. In particular, the dimension of the special spaces of Γ is then $i+1$.

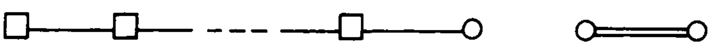
In view of the results in [40], every special space and fat hyperline of Γ meeting in a line determine a convex subgeometry of type $A_{1+2,2}$. We denote the set of all these Grassmannians by Γ_0, the set of all special spaces by Γ_1, $\Gamma_2 = \mathcal{P}$, and the set of all fat hyperlines by Γ_3. We also define an incidence between these objects as follows:

(i) an object of type 0 (resp., 1) is incident with one of type 3 if they intersect in plane (resp., line);

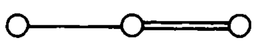
(ii) inclusion for all remaining cases.

We observe the following properties in Γ:
(i) every line is contained in exactly one special space;
(ii) every plane of a fat hyperline is contained in exactly one member of Γ_0. (Consider an arbitrary line in this plane α; then it is contained in the special space R. Both R and α determine a unique member of Γ_0.)

If we take $1+3=n$, then we claim that we defined a geometry of $\{4,\ldots,n\}$-truncated type C_n. For this we look first at the residue of a point p (object of type 2): we have the geometry of all Grassmannians, special spaces and fat hyperlines containing p. Otherwise stated, the geometry of direct products of type $A_i \times A_1$, special spaces (type A_i) and fat quads (type C_2) of the residue $\Gamma_p \simeq A_i \times C_2$. This gives us a truncated geometry of type:



On the other hand, we consider the residue of an object of type 3 (a fat hyperline), say H. In view of the properties (i) and (ii) stated above we can identify this geometry with the geometry of planes, lines and points lying in H. Therefore that is a diagram of type:



From this, the residual connectedness also follows, proving our claim.

Now we can apply Theorem 2 from [2]: Γ is a $\{4,\ldots,n\}$-truncated geometry Δ/A, with Δ a building of type C_n and A a group of automorphisms of Δ. Since every line is contained in one special space and one fat hyperline, the lines of Γ can be identified with the 2-shadow of flags of type $\{1,3\}$. Renumbering the nodes as usual we conclude that Γ is the geometry of points and lines of a building of type $C_{n,n-2}$ ($n \geq 4$) or a quotient of it.

3. Polarized Spaces That are Locally a Direct Product

In this section we are considering a polarized space such that through every line of it go exactly three max spaces. We start with the study of the residue Γ_p which is connected in view of axiom (3) (i). Further, every point x of Γ_p is contained in exactly three max spaces, and all quads on x are grids. We introduce the following notation for the unique member v of $a^\perp \cap b^\perp - \{u\}$ (for $u \in a^\perp \cap b^\perp$): $v = a + b - u$.

Lemma 3.1. *Let X and x' be two collinear but distinct points of Γ_p. Let a and b be two points collinear with x but not with x', $a' = x' + a - x$, $b' = x' + b - x$. Then $a \perp b$ iff $a' \perp b'$.*

Proof. Suppose first that a, b and x are on the same line; then the grid on x, x' and a contains b, and also the line a'/x'. This line goes through b', so this case is settled.

Next, assume that a and b are collinear, but the line ab does not contain the point x. In order to derive a contradiction suppose that a' and b' are not collinear. Call the max space through the line $x'b'$, R'. We show that we can assume that a is collinear with a point of R'. Suppose the contrary. The point b' is contained in three max spaces, namely one through the line bb', R' and a third, say, R''. Then the point $a'' = a + b' - b$ must belong to R''. Consider a point c on the line ax, distinct from a and x. If R' has a point c' collinear with c, then we interchange the roles of a and x (and derive a contradiction, as will be proved later on). So assume again that $c'' = c + b' - b$ belongs to R'', then $a'' \perp c''$. If Q is the grid on a, c, c'' and a'' (If $c \perp a''$ then $x \perp a''$ and so $a'' = x'$, a contradiction) then Q contains the line $ca = ax$, and hence $x^{\perp} \cap R'' \notin \emptyset$. But this is impossible because $x^{\perp} \cap b'^{\perp} = \{b, x'\}$. Hence a is collinear with a point y of R'. This implies that $a^{\perp} \cap x'^{\perp}$ would contain three distinct points a', y and x, another contradiction. We conclude that a' and b' are collinear.

On the other hand, as $a = x' + a' - x$ and $b = x' + b - x$ we can apply the first part of this proof again to show the converse. □

LEMMA 3.2. *The residue Γ_p can be identified with the direct product of three projective spaces, and so is of type $A_i \times A_j \times A_k$.*

PROOF. The first remark that there is one max space on every line of Γ_p. We prove now that two max spaces R_1 and R_2 meeting in a point x, are contained in just one convex subgeometry of Γ_p that can be identified with $R_1 \times R_2$.

Consider a point a_1 (resp., a_2) distinct of x in R_1 (resp., R_2). Write

$$\theta(a_1, a_2) = a_1 + a_2 - x, \qquad \theta(a_1, x) = a_1,$$
$$\theta(x, a_2) = a_2, \qquad \theta(x, x) = x,$$
$$D = \{\theta(a_1, a_2) \mid a_1 \in R_1, a_2 \in R_2\}.$$

The set D is a subset of the convex hull of R_1 and R_2. We claim that the map θ is an isomporphism:

$$\theta\colon R_1 \times R_2 \to D$$
$$(a_1, a_2) \to \theta(a_1, a_2).$$

Because $x^{\perp} \cap \theta(a_1, a_2) = \{a_1, a_2\}$, it follows from $\theta(a_1, a_2) = \theta(b_1, b_2)$ that $a_1 = b_1$ and $a_2 = b_2$. So θ is a bijection. We show that θ^{-1} is an isomorphism.

Consider therefore a point b_1 in $R_1 - a_1x$ and b_2 in $R_2 - a_2x$. Then we prove that $\theta(a_1, a_2)$ and $\theta(b_1, b_2)$ are not collinear by deriving a contradiction. Thus we assume that $\theta(a_1, a_2)$ and $\theta(b_1, b_2)$ are collinear. From the foregoing lemma it follows that $\theta(a_1, b_2)$ is collinear with $\theta(a_1, a_2)$ and with $\theta(b_1, b_2)$. If these three distinct points are on the same line, then $b_2 \perp \theta(b_1, b_2)$ and $b_2 \perp \theta(a_1, b_2)$ implies $b_2 \perp \theta(a_1, a_2)$ by axiom (1). But then $b_2 = a_2$. So these three points should not be one the same line. Now a_1, $\theta(b_1, b_2)$ belongs to $\theta(a_1, b_2)^{\perp} \cap \theta(a_1, a_2)^{\perp}$, so they are collinear. This means that $a_1 = b_1$, a final contradiction.

From this we derive that if $\theta(a_1, a_2) \perp \theta(b_1, b_2)$, then $a_1 = b_1$ or $a_2 = b_2$. Suppose we are in the first case, with a_1, a_2, b_2 distinct from x. Let L be the line joining $\theta(a_1, a_2)$ and $\theta(a_1, b_2)$. For an arbitrary point c of the line a_2b_2, $\theta(a_1, c)$ belongs to L (indeed, the grid on b_2 and $\theta(a_1, a_2)$ contains a_2 and $\theta(a_1, b_2)$ and hence also the lines a_2b_2 and L. Therefore $\theta(a_1, c)$ is the unique point in $c^{\perp} \cap L$). An analogous reasoning holds for the latter case in which $a_2 = b_2$.

Now consider two distinct points a_1, b_1 in R_1 (take $b_1 \neq x$) and a point b_2 in R_2 distinct from x. Since $x^{\perp} \cap \theta(b_1, b_2)^{\perp} = \{b_1, b_2\}$, $a_1 \not\perp \theta(b_1, b_2)$. It is also clear that $b_1 \not\perp b_2$. All this leads to the fact that the set D is a subspace of Γ_p and θ is an isomorphism. Hence the diameter of D is 2, and D is also 2-convex.

We describe D somewhat further: the max spaces contained in D split up into two families, $\mathbb{R}_1$ and $\mathbb{R}_2$. Each family is a partition of D. If $R_1 \in \mathbb{R}_1$ and $R_2 \in \mathbb{R}_2$ then we note that $D = R_1 * R_2$, and 'par abus de langage' talk about the direct product of R_1 and R_2.

Let R_3 be the third max space on x. If X_k is an arbitrary point of R_3, then we can define $D_k = R_1^k * R_2^k$, wth R_1^k and R_2^k the two remaining max spaces on x distinct from R_3.

We prove that D_i and D_j are disjoint for $i \neq j$. To derive a contradiction, suppose y belongs to D_i and D_j. If $d(y, R_3) = 1$, say $x' \in y^\perp \cap R_3$, then $y \perp x_i$ and $y \perp x_j$. (For $d(y, x_i) = 2$ would imply that x' belongs to D_i and D_j.) If $d(y, R_3) = 2$, take $x_i^\perp \cap y^\perp = \{a_1, a_2\}$ and $x_j^\perp \cap y^\perp = \{b_1, b_2\}$. In view of axiom (4) we can assume, without loss of generality, that $a_1 \perp b_1$. So a_1, b_1, x_j, x_i form a quadrangle having a_1 and x_1 in common with the quadrangle a_1, y, a_2, x_i. As $b_1 \perp y$, we have $a_2 \perp x_j$ if we apply Lemma 3.1 on x_i and a_1. Since a_2 is collinear with two distinct points x_i and x_j of R_3, it must belong to R_3. But then $d(y, R_3) = 1$, which is impossible.

Next we show that for any point z of $\Gamma_p - D$, $z^\perp \cap D$ contains exactly one point. In veiw of the connectedness of Γ_p, there is a point d of D at shortest distance of z. We have to prove that $d(z, d) = 1$. So suppose that $d = z_0, a_1, z_2, \ldots, z_n = z$ is a geodesic path from d to z, $n \geq 2$. In particular, $z_0 \not\perp z_2$, so we have grid on z_0 and z_2. This grid has a line L in common with D, for there are only three max spaces on z_0. But then $z_2^\perp \cap L \neq \emptyset$, and *a fortiori* $z_2^\perp \cap D \neq \emptyset$, a contradiction. We conclude that $d(z, D) \leq 1$. On the other hand, suppose that $d_1, d_2 \in z^\perp \cap D$ are two distinct points; then we can consider two cases. First, d_1 and d_2 are collinear. The line $d_1 d_2$ belongs to D, and in particular z is a point of D for D contains also the max space on $d_1 d_2$. Second, d_1 and d_2 are not collinear, but then $z \in d_1^\perp \cap d_2^\perp$ is again a point of D, for D is convex. The point $d \in z^\perp \cap D$ is unique. If $z \in D$, we take $z = d$. We define a map σ as follows: $\sigma(z) = d$.

We now show that for any point z of Γ_p, $d(z, R_3) \leq 2$. For this, let z be a point of Γ_p with $d(z, R_3) \geq 2$. From the foregoing sections follows the existence of a point $d \in z^\perp \cap D$. Of course, $d \not\perp x$by assumption, so take $x^\perp \cap d^\perp = \{a, b\}$. Define $c = z + a - d$ and $x' = c + x - a$. We claim that x' belongs to R_3. Suppose the contrary: then xx', xa and xb are three lines on x of which no two are contained in the same projective space. Together with R_3, there would be at least four distinct max spaces on x. This contradiction proves our claim.

Next we prove that there is exactly one subspace D_k on any given point z of Γ_p. In view of what we have already proved, it suffices to show that there is at least one on a given point z. First, suppose that $d(z, R_3) = 2$, and choose x_k in R_3 such that $d(z, x_k) = 2$. The points z and x_k are contained in a unique quad Q_k. Of course, this quad has no lines in common with R_3 for $d(z, R_3) = 2$. Both lines on x_k of Q_k are thus in D_k. In particular, $Q_k \subset D_k$, and so $x_k \in D_k$. Secondly, suppose $d(z, R_3) = 1$, then choose $x_k \in z^\perp \cap R_3$, and clearly D_k will contain z. Finally, if $z \in R_3$, then take $z = x_k$. With the notations introduced here, we define a map τ as $\tau(z) = x_k$.

As a last step in this proof, we define the following canonical map:

$$\rho: \Gamma_p \to R_3 \times D, \qquad z \to (\tau(z), \sigma(z)).$$

We will prove that ρ is an isomorphism.

First of all, ρ is a bijection: for a point x of R_3 and a point d of D, we construct a point z of Γ_p for which $\tau(z) = x$ and $\sigma(z) = d$. So consider the subspace D_k on x. In the very same way as before, we can show that $d^\perp \cap D_k$ contains exactly one point z. For this z clearly $\tau(z) = x$ and $\sigma(z) = d$ for $z \perp d$.

We make the following observations for an arbitrary point z of Γ_p. If D_k is the subspace on z described above, there is exactly one max space $R(z)$ on z, not contained in D_k. It should be clear by now that $R(z)$ meets every D_k in one point, and that the set of all such max spaces (in particular R_3) constitute a partition of Γ_p. Let L be any line of Γ_p, and y a point of L. Then there are two (non-trivial) cases that can occur.

(1) L is in a subspace D_k ($\neq D$) on y. Let z be any point distinct from y on L. Through z and $\sigma(y)$ goes a quad Q. This quad meets D in a line L' on $\sigma(y)$. There is a point z' on L' collinear with z. Since $z' \in D$ and only one point of D is collinear with z (if $z \notin D$), we have $z' = \sigma(z)$. Hence,

$$\{\sigma(z) \mid z \in L\} = L'$$

is a line of D, so $\rho(L) = \{\tau(y)\} \times L'$.

(2) L is a max space $R(y)$ ($\neq R_3$) on y; choose a point $z(\neq y)$ on L.
If $y \perp \tau(y)$, consider the quad on z and $\tau(y)$. It meets R_3 in a line L' on $\tau(y)$. Call z' the point of L' collinear with z; then $z' = \tau(z)$. In this case we also have

$$\{\tau(z) \mid z \in L\} = L',$$

so $\rho(L) = L' \times \{\sigma(y)\}$. If, on the contrary, $y \not\perp \tau(y)$, choose $a \in y^\perp \cap \tau(y)^\perp$. The quad on a and z has a line L' on a in common with $R(a)$. Let $z' \in z^\perp \cap L'$. The quad on $\tau(y)$ and z' meets R_3 in a line L'' on $\tau(y)$, so we can take $z'' \in z'^\perp \cap L''$. It is easy to see that if $d(z, R_3) = 2$, then $d(y, R_3) = 2$: $z \in R_3$ is excluded at once for, otherwise, $d(y, R_3) = 1$, and $d(z, R_3) = 1$ implies $y \perp \tau(z)$ (interchange the roles of y and z in the first part of this case), a contradiction. But z'' is a point of R_3 at distance 2 of z, so $z'' = \tau(z)$. In particular there holds

$$\{\tau(z) \mid z \in L\} = L'',$$

for when z runs through the line L, then z' will run through L' and hence z' through L'' by applying the first part of this second case. The constructed L'' is therefore independent of z and hence $\rho(L) = L'' \times \{\sigma(y)\}$.

All this together shows us that ρ is an isomorphism. Both parts of this proof lead us to the conclusion that

$$\pi: \Gamma_p \to R_1 \times R_2 \times R_3$$
$$z \to (z_1, z_2, \tau(z))$$

with

$$\sigma(z) = \theta(z_1, z_2)$$

is an isomorphism. The lemma follows. □

COROLLARY 3.3. *Suppose that there are exactly three max spaces on each line of the polarized space Γ, then Γ is locally $A_i \times A_j \times A_k$.*

PROOF. It suffices, in view of the theorem just proved, to show that the dimensions i, j, k are independent of the choice of the point p. Since Γ is connected, we can restrict our discussion to the case of two collinear points p and q. In Γ_p the line pq is a point, so there are three max spaces on pq of dimensions i, j and k. In Γ this means that the line pq is contained in three max spaces od dimensions $i+1$, $j+1$ and $k+1$. Back in Γ_q, qp is a point contained in three max spaces of dimensions i, j and k in Γ_q. □

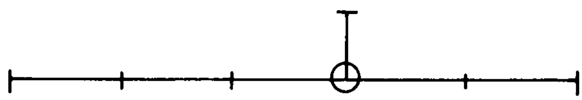
4. CHARACTERIZATION OF $E_{7,4}$ AND $E_{8,5}$

Suppose that every line X of the polarized space Γ is contained in exactly one maximal plane, one maximal 3-space and one 4-space (resp., 5-space). Then Γ is the geometry of points and lines of a building of type $E_{7,4}$ (resp., $E_{8,5}$), or a quotient of it.

PROOF. Just as in the foregoing theorem we can apply the results from [ ]. There follows the existence of the following objects:
(i) Convex subspaces isomorphic to a Grassmannian $E_{4,2}$, containing maximal planes and maximal 3-spaces. We call this set Γ_1.
(ii) Convex subspaces isomorphic to a Grassmannian $A_{5,2}$, containing maximal planes and 4-spaces. We call this set Γ_2.
(iii) Subspaces isomorphic to a Grassmannian $A_{6,3}$, containing maximal 3-spaces and 4-spaces. We call this set Γ_3.
Conversely, given a maximal plane (resp., 3-space, 4-space) and a maximal 3-space (resp., 4-space, plane) intersecting in a linek then there is a unique member of Γ_1 (resp., Γ_3, Γ_2) containing both.

Moreover, take $\Gamma_0 = \mathscr{P}$ and define the following incidence: (i) a point is incident with a Grassmannian (i.e. a member of Γ_1, Γ_2 or Γ_3) if it lies in it; (ii) two Grassmannians are incident if they meet in a maximal singular subspace.

We now check that we have the following truncated diagram:

$E_{7,4}$:  (*)

Remark first that on every maximal plane of Γ there is exactly one member of Γ_1 and one of Γ_2. Analogously, there is unique member of Γ_2 and one of Γ_3 (resp. Γ_1) on every maximal 3-space of Γ. Consider now a 1-variety G_1 then the residual geometry of G_1 consists of the set of all points of G_1, all 2-varieties meeting G_1 in a maximal plane and all 3-varieties meeting G_1 in a maximal 3-space. In view of both remarks made above, we can identify this geometry with the points of G_1, the maximal planes and maximal 3-spaces lying in G_1. Incidence between a maximal plane and a maximal 3-space becomes meeting in a line of G_1. Therefore we obtain a truncated geometry of type $A_{4,2}$.

In particular it follows that the residues of flags of rank 2 containing a 1-variety are truncated geometries of the type as predicted by the truncated diagram (*). Moreover, these residues are residual connected. In the same way we can examine a 2-variety G_2, checking with this all remaining conditions needed to apply Theorem 4 of [2]. It is easily verified that the lines correspond to the flags of type $\{1, 2, 3\}$ and conversely.

Taking account of the varieties we called points, we have that Γ is a truncated geometry of type $E_{7,4}$. In the same way one can prove the analogous statement for $E_{8,5}$. The theorem follows. □

5. CHARACTERIZATION OF $D_{n,n-2}$ $(n \geqslant 5)$

Suppose that every line X of the polarized space Γ is contained in exactly two maximal planes and one $(j+1)$-space $(j+1 \geqslant 3)$. Then Γ is the geometry of points and lines of a building of type $D_{n,n-2}$ for $n \geqslant 5$, or a quotient of it.

PROOF. It follows from Corollary 3.3 that Γ is locally $A_1 \times A_1 \times A_j$. Hence, Γ satisfies the conditions described in 2, [10]. So there exists a set Γ_0 of convex subspaces of Γ, such that the following properties hold: (i) each member of Γ_0 is isomorphic to a Grassmannian $A_{j+2,2}$; (ii) every maximal plane and $(j+1)$-space meeting in a line generate a member of Γ_0.
We also define the following sets: Γ_1 is the set of all $(j+1)$-spaces; $\Gamma_2 = \mathscr{P}$. Γ_3 is the set of all hyperlines not contained in a member of Γ_0, and the corresponding incidence: (i)

an object of type 0 (resp. type 1) and one of type 3 are incident if they meet in a plane (resp. a line); (ii) inclusion for all remaining cases.

In the same way as in the proof of Theorem, one can check that this defines a $\{4, \ldots, n\}$-truncated geometry Δ/A, with Δ a building of type C_n and A a group of automorphisms of Δ $(n = j + 3 \geqslant 5)$.

We remark that every 1-variety is contained in two 0-varieties. Then, by [12], Δ can also be seen as a building of type D_n. Hence, Γ is the geometry of points and lines of a building of type $D_{n,n-2}$ for $n \geqslant 5$, or a quotient of it by a group of automorphisms (not necessarily preserving the types n and $n-1$). □

6. Characterization of $D_{4,2}$

Suppose that every line X of the polarized space Γ is contained in exactly three planes. Then Γ is the geometry of points and lines of a building of type $D_{4,2}$.

Proof. By Lemma 3.2, is locally $A_1 \times A_1 \times A_1$. Furthermore, every line is contained in exactly three lean hyperlines of rank 3. Now take: $\Gamma_1 = \mathcal{P}$; $\Gamma_2 = \mathcal{L}$; Γ_3 is the set of all planes of Γ; Γ_4 is the set of all hyperlines of Γ; together with inclusion as an incidence relation. Then we obtain a metasymplectic space where every line is contained in three hyperlines. By Tits [12], it follows that we may consider Γ as the geometry of points and lines of a building of type $D_{4,2}$ (because this geometry has diameter 3, quotients cannot occur). □

Acknowledgements

The author is supported by the N.F.W.O. (National grant for scientific research of Belgium).

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Received 28 June 1985 and in revised form 5 June 1987

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